J. Math. Anal. Appl. 427 (2015) 581–599

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Weighted shifts induced by Hamburger moment sequences

George R. Exner [a], Joo Young Jin [b], Il Bong Jung [b,*], Mi Ryeong Lee [c]

[a] Department of Mathematics, Bucknell University, Lewisburg, PA 17837, USA
[b] Department of Mathematics, Kyungpook National University, Daegu 702-701, Republic of Korea
[c] Institute of Liberal Education, Catholic University of Daegu, Gyeongsan, Gyeongbuk 712-702, Republic of Korea

ARTICLE INFO

Article history:
Received 28 August 2014
Available online 18 February 2015
Submitted by L. Fialkow

Keywords:
Weighted shift
Hamburger measure
Subnormality
Hyponormality
Completion problem

ABSTRACT

We indicate how our subject emerges from the confluence of several streams of analysis, including the classical moment problems, the theory of positive matrices and subnormal operator theory. Some new properties $H(n)$ ($n = 1, 2, \ldots$) and a Hamburger-type weighted shift are considered via a Hamburger moment sequence. We discuss examples to show the various $H(n)$ are distinct; study flatness, backward n-step extensions and perturbations of weighted shifts; and, given three initial weights $\alpha_0, \alpha_1, \alpha_2$ with $\alpha_0 \le \alpha_2 < \alpha_1$, we produce a completion: a weighted shift of Hamburger type but not subnormal, extending a (subnormal) completion by Stampfli in the case $\alpha_0 < \alpha_1 < \alpha_2$.

1. Introduction and preliminaries

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. We denote by $[A, B] := AB - BA$ the *commutator* of A and B in $\mathcal{L}(\mathcal{H})$. Let $\mathbb{N}$ [resp., $\mathbb{Z}_+$] be the set of positive integers [resp., nonnegative integers]. We write $\mathbb{R}$ [resp., $\mathbb{R}_+, \mathbb{C}$] for the set of real [resp. nonnegative real, complex] numbers and let $\mathbb{R}_+^0 := \mathbb{R}_+ \backslash \{0\}$.

An operator T in $\mathcal{L}(\mathcal{H})$ is *subnormal* if it is (unitarily equivalent to) the restriction of a normal operator to an invariant subspace, and *hyponormal* if $[T^*, T] \ge 0$. It is well-known that an operator T in $\mathcal{L}(\mathcal{H})$ is subnormal if and only if $\sum_{0 \le i,j \le n} \langle T^{*i} T^j h_i, h_j \rangle \ge 0$ for all $h_i, h_j \in \mathcal{H}$ and $n \in \mathbb{N}$ [4,11]. For a fixed $n \in \mathbb{N}$, an operator $T \in \mathcal{L}(\mathcal{H})$ is *n-hyponormal* if $\sum_{0 \le i,j \le n} \langle T^{*i} T^j h_i, h_j \rangle \ge 0$ for all $h_i, h_j \in \mathcal{H}$. Thus $T \in \mathcal{L}(\mathcal{H})$ is subnormal if and only if T is n-hyponormal for all $n \in \mathbb{N}$. Obviously, the implications "subnormal $\Rightarrow \cdots \Rightarrow$ 2-hyponormal $\Rightarrow$ hyponormal" hold, and it is well-known that each converse is not always true [5,12].

* Corresponding author.
E-mail addresses: exner@bucknell.edu (G.R. Exner), pss9611@knu.ac.kr (J.Y. Jin), ibjung@knu.ac.kr (I.B. Jung), leemr@cu.ac.kr (M.R. Lee).

http://dx.doi.org/10.1016/j.jmaa.2015.02.033
0022-247X/

In the study of these classes, the weighted shifts W_α with weight sequence $\alpha = \{\alpha_i\}_{i=0}^\infty$ in $\mathbb{R}_+^0$ have played an important role. There are several standard questions and approaches to study the structure of weighted shifts W_α, such as flatness, backward extension of a weight sequence to a new one, completion of an initial segment of weights to a weight sequence, etc. Concerning flatness, J. Stampfli [17] proved that if W_α is subnormal with $\alpha_n = \alpha_{n+1}$ ($n \in \mathbb{Z}_+$), then $\alpha_1 = \alpha_2 = \cdots$. His result about flatness in subnormal weighted shifts was improved to the case of 2-hyponormality in [5]. The subnormality of a weighted shift W_α is related closely to the Stieltjes moment problem which we describe next.

Given a sequence $\{\gamma_n\}_{n=0}^\infty \subset \mathbb{R}_+^0$, *the Stieltjes moment problem* entails determining whether there exists, and finding when it does, a positive Borel measure μ on $\mathbb{R}$ supported on $\mathbb{R}_+$ such that

$$\gamma_n = \int\limits_{\mathbb{R}_+} t^n \, d\mu(t), \quad n \in \mathbb{Z}_+.$$

Such a sequence $\{\gamma_n\}_{n=0}^\infty$ [resp., measure μ] is called a *Stieltjes moment sequence* [resp., *Stieltjes moment measure*]. Furthermore, it is well-known that $\{\gamma_n\}_{n=0}^\infty$ is a Stieltjes moment sequence if and only if the two infinite matrices $(\gamma_{i+j})_{0\le i,j<\infty}$ and $(\gamma_{i+j+1})_{0\le i,j<\infty}$ are positive (cf. [16]). (We mean by this slight – and common – abuse of language that each of the principal submatrices of $(\gamma_{i+j})_{0\le i,j<\infty}$ and $(\gamma_{i+j+1})_{0\le i,j<\infty}$ is non-negative.)

Given a sequence $\{\gamma_n\}_{n=0}^\infty \subset \mathbb{R}$, the analogous *Hamburger moment problem* relaxes the requirement to a positive Borel measure μ supported merely on $\mathbb{R}$ such that

$$\gamma_n = \int\limits_{\mathbb{R}} t^n \, d\mu(t), \quad n \in \mathbb{Z}_+.$$

If this is possible the sequence $\{\gamma_n\}_{n=0}^\infty$ and measure μ are called a *Hamburger moment sequence* and a *Hamburger moment measure*, respectively. It follows from [16] that $\{\gamma_n\}_{n=0}^\infty$ is a Hamburger moment sequence if and only if $(\gamma_{i+j})_{0\le i,j<\infty}$ is positive.

We set some notation for the standard testing ground of weighted shift operators. Let $\{e_i\}_{i\in\mathbb{Z}_+}$ be the canonical orthonormal basis for $\ell^2(\mathbb{Z}_+)$. Given a weight sequence $\alpha = \{\alpha_k\}_{k=0}^\infty$ of positive real numbers, we define the weighted shift W_α by $W_\alpha e_k = \alpha_k e_{k+1}$ and extend by linearity. (Observe that if the sequence $\{\alpha_k\}_{k=0}^\infty$ is bounded then W_α is a bounded operator.) We define the moment sequence $\{\gamma_i\}_{i=0}^\infty$ by

$$\gamma_0 = 1; \; \gamma_i := \alpha_0^2 \cdots \alpha_{i-1}^2, \quad i \in \mathbb{N}.$$

Occasionally, when more than one shift is in play, we will use notation like "$\gamma_n(W_\alpha)$" in the obvious sense.

We adopt, here and in what follows, the convention that we insist that the weights are non-negative (in fact, almost always positive); note that it is shown in [15] that for any property of interest preserved by unitary equivalence this restriction is without loss of generality. It is well-known that W_α is subnormal if and only if there is a (unique) positive Borel measure μ supported on $[0, \|W_\alpha\|^2]$ such that

$$\gamma_i = \int\limits_{\mathbb{R}_+} t^i \, d\mu(t), \quad i \in \mathbb{Z}_+,$$

which is as above equivalent to the positivity of the usual two infinite matrices of moments $(\gamma_{i+j})_{0\le i,j<\infty}$ and $(\gamma_{i+j+1})_{0\le i,j<\infty}$. (The obvious question of when such a measure exists, supported on a finite interval in $\mathbb{R}_+$, for a sequence $\{\gamma_n\}_{n=0}^\infty$ is the classical *Hausdorff moment problem*, with attendant Hausdorff measure, Hausdorff sequence, and the necessary and sufficient condition just given.) The resulting measure μ is called the *Berger measure* (for W_α).

Note that the subnormality of W_α is related to positivity of both of the infinite matrices $(\gamma_{i+j})_{0\le i,j<\infty}$, $(\gamma_{i+j+1})_{0\le i,j<\infty}$, where the positivity of the first guarantees that $\{\gamma_n\}_{n=0}^{\infty}$ is a Hamburger moment sequence and the additional positivity of the second matrix promotes the sequence to a Stieltjes (and in the bounded case, Hausdorff) moment sequence (and subnormality results). In this paper, we consider what may be obtained from merely positivity of one or the other of the matrices, or principal submatrices of bounded size, for the standard matters of flatness, backward extensions, and completion problems for sequences and weighted shifts. For this purpose, we define properties $H(n)$ and $\widetilde{H}(n)$ and discuss some operator properties related to them.

The organization of this paper is as follows. In Section 2, we give basic definitions, constructions, and examples. In Section 3, we discuss relationships among subnormality, Hamburger-type property, properties $H(n)$ and $\widetilde{H}(n)$, and obtain some results distinguishing the various classes under study. In Section 4, we consider flatness (the propagation of equal adjacent weights to some or all other weights) and in Section 5 we consider matters of backward n-step extensions and perturbations. In Section 6 we consider completion problems (indicating, for example how to complete three initial weights and when the resulting completion results in a shift with positive weights) and finally give a remark.

For a subset $\mathcal{E}$ of a Hilbert space $\mathcal{H}$, we denote by $\vee\mathcal{E}$ the closed linear span of $\mathcal{E}$ in $\mathcal{H}$. And we let δ_p denote the Dirac point mass measure at p throughout this paper. Some of the calculations in this paper were aided by use of the software tool Mathematica (see [13]).

2. Basic constructions

Let $\alpha = \{\alpha_k\}_{k=0}^{\infty}$ be a sequence of positive real numbers and let W_α be the associated weighted shift with weight sequence α. For $k, n \in \mathbb{Z}_+$, we set

$$M_n(k) = \begin{pmatrix} \gamma_k & \gamma_{k+1} & \cdots & \gamma_{k+n} \\ \gamma_{k+1} & \gamma_{k+2} & \cdots & \gamma_{k+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{k+n} & \gamma_{k+n+1} & \cdots & \gamma_{k+2n} \end{pmatrix}.$$

Note that the matrix is of size $(n+1)$ by $(n+1)$, and is, in fact, the standard matrix considered for n-hyponormality of weighted shifts (it is Theorem 4 of [5] that n-hyponormality of a weighted shift is equivalent to non-negativity of $M_n(k)$ for all $k \in \mathbb{Z}_+$).

Definition 2.1. A weighted shift W_α has *property* $H(n)$ [resp., *property* $\widetilde{H}(n)$] if $M_n(k) \ge 0$ for all $k = 0, 2, 4, \ldots$ [resp., $M_n(k) \ge 0$ for all $k = 1, 3, 5, \ldots$]. And W_α has *property* $H(\infty)$ [resp., property $\widetilde{H}(\infty)$] if it has property $H(n)$ [resp., property $\widetilde{H}(n)$] for all $n \in \mathbb{N}$. In particular, we say that W_α is a *Hamburger-type weighted shift* if W_α has property $H(\infty)$.

Note that, for some $n \in \mathbb{N}$, W_α is n-hyponormal if and only if W_α has both properties $H(n)$ and $\widetilde{H}(n)$. Therefore W_α is subnormal if and only if it has properties $H(n)$ and $\widetilde{H}(n)$ for all $n \in \mathbb{N}$. Moreover, elementary computations show that W_α has property $H(1)$ [resp., property $\widetilde{H}(1)$] if and only if $\alpha_{2n+1} \ge \alpha_{2n}$ [resp., $\alpha_{2n+2} \ge \alpha_{2n+1}$] for all $n \in \mathbb{Z}_+$. Obviously, then, the properties $H(n)$ and $\widetilde{H}(n)$ are distinct for each n and distinct from n-hyponormality, but note that the well-known fact that W_α is hyponormal (which is 1-hyponormal) if and only if its weights are weakly increasing splits neatly into two requirements related to the properties $H(1)$ and $\widetilde{H}(1)$. It turns out that, unsurprisingly, even property $H(\infty)$ does not imply either $\widetilde{H}(n)$ or n-hyponormality for any n (see Example 2.2).

We emphasize the fact that if W_α is Hamburger-type then the sequence $\{\gamma_n\}_{n=0}^{\infty} \ge 0$ is a Hamburger moment sequence, but under our convention of positive weights it carries the additional information that each γ_n is positive. If W_α is Hamburger-type we will sometimes call the measure associated to W_α the *Hamburger measure* μ.

We turn to some examples showing certain classes are distinct.

Example 2.2. Consider $\alpha : \alpha_n = \sqrt{\frac{2^{n+1}+(-1)^{n+1}}{2^n+(-1)^n}}$ $(n \geq 0)$. Observe that the measure $\mu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_2$ satisfies

$$\gamma_n = \frac{1}{2}(2^n + (-1)^n) = \int_{\mathbb{R}} t^n d\mu(t), \quad n \in \mathbb{Z}_+.$$

Hence W_α has property $H(\infty)$. But since $\det(\gamma_{i+j+1})_{i,j=0}^1 = -\frac{9}{2} < 0$, W_α does not have property $\widetilde{H}(n)$ for any $n \in \mathbb{N}$. So W_α is not n-hyponormal for any $n \in \mathbb{N}$. This example shows that in general the properties $H(\infty)$ and $\widetilde{H}(n)$ (and thus certainly $\widetilde{H}(\infty)$ and subnormality) are different.

In general, property $H(n)$ does not imply property $H(n+1)$ for any $n \in \mathbb{N}$.

Example 2.3. Let $\alpha : \sqrt{x}, \sqrt{\frac{k+1}{k+2}}$ $(k \geq 1)$ and let W_α be the associated weighted shift. By the techniques in the proof of [9, Th. 4] (and see originally [5, Prop. 7]), we obtain that W_α has property $H(n)$ if and only if $0 \leq x \leq \frac{(n+1)^2}{2n(n+2)}$ for $n \in \mathbb{N}$. (In fact, in this case, property $H(n)$ for W_α is equivalent to n-hyponormality and the sole new thing to check is that what is in play is the property $H(n)$ portion of n-hyponormality.)

Some improved examples related to properties $H(n)$, $\widetilde{H}(n)$ and n-hyponormality will be discussed in the next section.

We pause to record an easy fact motivated by the example below of a "backward 1-step extension."

Example 2.4. (Continued from Example 2.2.) Let

$$\alpha(x) : \alpha_0 = \sqrt{x},\ \alpha_n = \sqrt{\frac{2^{n+1} + (-1)^{n+1}}{2^n + (-1)^n}}, \quad n \geq 1,$$

where x is a positive real variable. Then a direct computation shows that

(i) $W_{\alpha(x)}$ has property $H(1)$ if and only if $0 < x \leq 5$,

(ii) $W_{\alpha(x)}$ has property $H(n)$ for some $n \geq 2$ [or, for all $n \geq 1$] if and only if $0 < x \leq \frac{1}{2}$, which is equivalent to $W_{\alpha(x)}$ has property $H(\infty)$.

Recall that given a weight sequence $\alpha = \{\alpha_k\}_{k=0}^\infty$ (or weighted shift W_α), and given positive $x_m, x_{m-1}, \ldots, x_1$ we may define the backward m-step extension $\alpha(x_m, x_{m-1}, \ldots, x_1)$ by $\alpha(x_m, x_{m-1}, \ldots, x_1) = x_m, x_{m-1}, \ldots, x_1, \alpha_0, \alpha_1, \ldots$. (Equivalently, given the shift W_α, we may define a new shift $W_{\alpha(x_m, x_{m-1}, \ldots, x_1)}$ in the obvious way.) We will consider such backward m-step extensions further in Section 5, but for now we note that it is easy to see (by considering the matrices) that if W_α has some property $H(n)$, then a backward 1-step extension of W_α has property $\widetilde{H}(n)$.

It is worthwhile to mention briefly the matter of uniqueness for the measure associated with a Hamburger moment sequence.

Remark 2.5. In fact, it is well known that the Hamburger moment measure need not be unique (often called "indeterminacy"). However, if $\{\gamma_n\}_{n=0}^\infty$ is such that there exist C and D such that $|\gamma_n| \leq C \cdot D^n \cdot n!$ for all $n \in \mathbb{Z}_+$, the associate measure μ is unique (see [14, p. 205]). (In fact, there is a yet more general sufficient condition due to Carleman (see [2]).) Since we consider bounded shifts, we have $\alpha_n \leq \|W_\alpha\|^2 =: K$, and clearly $\gamma_n \leq K^n$ for all n, and thus any solution, if it exists, is unique. It may be of interest to consider unbounded densely defined shifts, or some of these questions in the indeterminate case, but we do not consider such matters here.

3. Distinctions

Recall the determinant of the Cauchy matrix with (i,j) entry $\frac{1}{x_i+y_j}$ is

$$\frac{\prod_{0\le i<j\le n}(x_j-x_i)(y_j-y_i)}{\prod_{0\le i,j\le n}(x_i+y_j)}. \tag{3.1}$$

We now give a proposition showing that the properties $H(n)$, $\widetilde{H}(n)$ and n-hyponormality are distinct by using the Cauchy determinant formula (3.1). Denote the determinant of a matrix M by $|M|$.

Proposition 3.1. *Let $\alpha(x) : \sqrt{\frac{1}{2}}, \sqrt{x}, \left\{\sqrt{\frac{k+1}{k+2}}\right\}_{k=2}^{\infty}$ be a weight sequence where x is a positive real variable and let $W_{\alpha(x)}$ be the associated weighted shift with weight sequence $\alpha(x)$. Then, for $n \ge 2$,*

(i) *$W_{\alpha(x)}$ has property $H(n)$ if and only if $\delta_n^{(1)} \le x \le \delta_n^{(2)}$, where $\delta_n^{(1)}$ and $\delta_n^{(2)}$ are roots of $\Phi_n(x) = A_n x^2 + B_n x + C_n$ with*

$$\begin{aligned} A_n &:= -\tfrac{1}{4}(n+1)^{-2}\left(10n+9n^2+4n^3+n^4-8\right)(n+3)(n-1)n, \\ B_n &:= \tfrac{1}{3}(n+2)^{-1}\left(12n^3-8n^2-24n+13n^4+6n^5+n^6+24\right), \\ C_n &:= -\tfrac{1}{9}n^2(n+1)^2(n+2). \end{aligned}$$

(ii) *$W_{\alpha(x)}$ has property $\widetilde{H}(n)$ if and only if $0 < x \le \frac{2(n+1)^2(n+2)^2}{3(n^2+3n)(n^2+3n+4)} =: \delta_n^{(3)}$ (note that $0 < \delta_n^{(1)} < \delta_n^{(3)} < \delta_n^{(2)} (n \ge 2)$, $\delta_n^{(1)} \nearrow \frac{2}{3}$ and $\delta_n^{(2)} \searrow \frac{2}{3}$).*

(iii) *$W_{\alpha(x)}$ is n-hyponormal [resp., subnormal] if and only if $\delta_n^{(1)} \le x \le \delta_n^{(3)}$ [resp., $x = \frac{2}{3}$].*

Proof. (i) To consider the properties $H(n)$ of $W_{\alpha(x)}$, we study $M_n(0) = \frac{3x}{2}\Delta_n(x)$, where

$$\Delta_n := \Delta_n(x) = \begin{pmatrix} \frac{2}{3x} & \frac{1}{3x} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \frac{1}{3x} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+2} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+2} & \frac{1}{n+3} & \cdots & \frac{1}{2n+1} \end{pmatrix}.$$

By some elementary determinant operations, we may prove that

$$|\Delta_n| = \left(\frac{2}{3x}-1\right)|\Delta_n^{(1)}| + 2\left(-\frac{1}{3x}+\frac{1}{2}\right)|\Delta_n^{(2)}| + |\Delta_n^{(3)}| - \left(\frac{1}{3x}-\frac{1}{2}\right)^2 |\Delta_n^{(4)}|$$

with

$$\Delta_n^{(1)} = \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+2} \\ \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+3} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+2} & \frac{1}{n+3} & \cdots & \frac{1}{2n+1} \end{pmatrix}, \qquad \Delta_n^{(2)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \cdots & \frac{1}{n+2} \\ \frac{1}{3} & \frac{1}{5} & \cdots & \frac{1}{n+3} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+3} & \cdots & \frac{1}{2n+1} \end{pmatrix},$$

$$\Delta_n^{(3)} = \begin{pmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n+1} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n+1} \end{pmatrix}, \qquad \Delta_n^{(4)} = \begin{pmatrix} \frac{1}{5} & \frac{1}{6} & \cdots & \frac{1}{n+3} \\ \frac{1}{6} & \frac{1}{7} & \cdots & \frac{1}{n+4} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+3} & \frac{1}{n+4} & \cdots & \frac{1}{2n+1} \end{pmatrix},$$

where $\Delta_n^{(i)}$, $i = 1, 3, 4$, are Cauchy matrices and $\Delta_n^{(2)}$ is a submatrix removing the second column and last row from a Cauchy matrix. It follows from a direct computation applying (3.1) that

$$|\Delta_n| = 12^2 n! \left(\prod_{k=4}^{n-1} k!\right)^3 \left(\prod_{k=n+3}^{2n+1} k!\right)^{-1} \frac{1}{x^2}\Phi_n(x),$$

where $\Phi_n(x)$ is the quadratic polynomial as in (i). (For an example of the method to compute a determinant for a matrix with an "omitted column," see the computations following equation (3.4) of [1].) For $x \in \mathbb{R}_+^0$ such that $\Phi_n(x) \geq 0$, we can check easily that $\Phi_k(x) > 0$, i.e., $|\Delta_k(x)| > 0$, $k = 2, 3, \ldots, n-1$. This allows us to use Sylvester's criterion (which is often called the *Nested Determinant Test*; for example, see [6, p. 213]) for positivity of a matrix: since the principle submatrices $\Delta_k(x)$ have strictly positive determinants, $\Delta_n \geq 0$ if and only if $|\Delta_n| \geq 0$, i.e., $\Phi_n(x) \geq 0$, which proves (i).

(ii) This case can be proved easily by following the same methods as for (i).

(iii) It follows from some direct computations that $0 < \delta_n^{(1)} < \delta_n^{(3)} < \delta_n^{(2)}$ for $n \geq 2$. The remaining parts are trivial. $\square$

For the next proposition we will use the observation that if $\binom{n}{k}$ is the usual binomial coefficient, then it is bounded above by n^k.

Proposition 3.2. *For any $n \in \mathbb{N}$, there exists a Hamburger-type weighted shift which is not subnormal but has property $\widetilde{H}(n)$.*

Proof. We will consider measures of the form

$$d\mu := \frac{1}{1+\epsilon}\left(\epsilon\delta_{-\epsilon} + \chi_{[0,1]}(t)dt\right), \quad 0 < \epsilon < 1, \tag{3.2}$$

where in fact ϵ will usually be close to zero.

We leave to the reader to show that if ϵ is sufficiently small then the moments γ_k are all positive, and assume henceforth without comment that we restrict to such ϵ. Note also that when we consider positivity of matrices of the γ_k of fixed size, since each γ_k includes the normalizing factor $\frac{1}{1+\epsilon}$ we may ignore it, and consider instead matrices with entries $\gamma_k' := (1+\epsilon)\gamma_k$.

Fix n a positive integer. To obtain property $\widetilde{H}(n)$, we must consider the positivity, for k odd, of matrices of the form

$$M_n'(k) = \begin{pmatrix} \gamma_k' & \gamma_{k+1}' & \cdots & \gamma_{k+n}' \\ \gamma_{k+1}' & \gamma_{k+2}' & \cdots & \gamma_{k+n+1}' \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{k+n}' & \gamma_{k+n+1}' & \cdots & \gamma_{k+2n}' \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{k+1} + \epsilon(-\epsilon)^k & \frac{1}{k+2} + \epsilon(-\epsilon)^{k+1} & \cdots & \frac{1}{k+n+1} + \epsilon(-\epsilon)^{k+n} \\ \frac{1}{k+2} + \epsilon(-\epsilon)^{k+1} & \frac{1}{k+3} + \epsilon(-\epsilon)^{k+2} & \cdots & \frac{1}{k+n+2} + \epsilon(-\epsilon)^{k+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{k+n+1} + \epsilon(-\epsilon)^{k+n} & \frac{1}{k+n+2} + \epsilon(-\epsilon)^{k+n+1} & \cdots & \frac{1}{k+2n+1} + \epsilon(-\epsilon)^{k+2n} \end{pmatrix}.$$

Denote by $C_n(k)$ the matrix resulting from the above with ϵ set to 0; of course this is one of the Hilbert submatrices and is known to be invertible (we will shortly give its inverse from [3]). Observe that we may write $M'_n(k) = C_n(k) + \mathbf{cr}$, where $\mathbf{r}$ is the row vector $\mathbf{r} = (-1, \epsilon, -\epsilon^2, \ldots, \epsilon(-\epsilon)^{n-1})$ and $\mathbf{c}$ is the column vector whose transpose is $\mathbf{c}^T = (\epsilon^{k+1}, -\epsilon^{k+2}, \ldots, (-\epsilon)^{k+n+1})$. It is then known that

$$\det M'_n(k) = \det C_n(k)(1 + \mathbf{r}(C_n(k))^{-1}\mathbf{c}),$$

which comes from Sylvester's determinant theorem (cf. [18]).

We will first show there is ϵ'_n such that for all $0 < \epsilon < \epsilon'_n$, the quantities $1 + \mathbf{r}C_n(k)^{-1}\mathbf{c}$ are positive for all $k \geq 1$. Citing [3], we have that the (i,j)th entry of $C_n(k)^{-1}$ is

$$(-1)^{i+j}(k+i+j+1)\binom{k+n+i+1}{n-j}\binom{k+n+j+1}{n-i}\binom{k+i+j}{i}\binom{k+i+j}{j}.$$

By the remark above the term $\binom{k+n+i+1}{n-j}$ is bounded above by $(k+n+i+1)^{n-j}$ which is clearly in turn bounded above by $(k+2n+1)^n$. Treating the other terms similarly, we have that each entry of $C_n(k)^{-1}$ is bounded above by $(k+2n+1)^{4n+1}$, and, in particular, is of order no higher than k^{4n+1} in k. It follows from a direct computation that

$$\|C_n(k)^{-1}\| \leq (n+1)(k+2n+1)^{4n+1}.$$

With obvious estimates for $\|\mathbf{r}\|$ and $\|\mathbf{c}\|$, we have

$$|\mathbf{r}C_n(k)^{-1}\mathbf{c}| \leq \epsilon^{k+1}(n+1)^2(2n+k+1)^{4n+1}.$$

But it is then elementary that for ϵ sufficiently small, we may ensure this quantity is less than 1 for all k, and so ensure that for some $\epsilon'_n > 0$ and for all $\epsilon < \epsilon'_n$, $\det M'_n(k)$ is strictly positive for all $k = 1, 2, \ldots$ (and then in particular for odd k).

Now our goal is positivity of the matrices $M'_n(k)$, and so far we have merely positivity of their determinants. To use the nested determinant test, we need as well positivity of principal submatrices. However, since we may perform the above analysis for each j, $1 \leq j \leq n$, yielding ϵ'_j, we finish by noting we have positivity of the $M'_n(k)$ for all k odd and all $0 < \epsilon < \epsilon_n := \min\{\epsilon'_1, \ldots, \epsilon'_n\}$, which is exactly property $\widetilde{H}(n)$ for such ϵ. $\square$

Using exact calculations with *Mathematica* [13] and some elementary calculus (which we do not reproduce here), one can show that with ϵ set to 1/2 in the above construction one obtains a Hamburger sequence of moments which are all positive but so that property $\widetilde{H}(1)$ does not hold (so the shift is not hyponormal); with ϵ set to 1/10 the shift has property $\widetilde{H}(1)$ by not $\widetilde{H}(2)$ (so it is hyponormal but not 2-hyponormal); with ϵ set to 88/1000 the shift has property $\widetilde{H}(2)$ by not $\widetilde{H}(3)$ (so it is 2-hyponormal but not 3-hyponormal). We conjecture that similar values of ϵ exist for this example to separate one "$H(\infty) + \widetilde{H}(n)$" from the next but cannot evaluate the relevant determinants in general.

We have the result analogous to that of Proposition 3.2 with the roles of property $H(n)$ and property $\widetilde{H}(n)$ reversed.

Proposition 3.3. *For any $n \in \mathbb{N}$, there exists a weighted shift W_α which is not subnormal but has property $\widetilde{H}(\infty)$ and property $H(n)$.*

Proof. Fix $n \in \mathbb{N}$ and consider the shift (Hamburger-type) W_α produced in the proof of Proposition 3.2 (for that same n). Let the weight sequence of W_α be $\alpha : \alpha_0, \alpha_1, \ldots$, and form a backward 1-step extension $\alpha(x)$

by prefixing a weight x. It is easy to see that the weighted shift $W_{\alpha(x)}$ has property $\widetilde{H}(\infty)$ and the only matrix whose positivity is in doubt for $H(n)$ is $M_n(0)(W_{\alpha(x)})$. Recall that in the course of the construction of W_α we ensured strict positivity (and positive determinant) of $M_{n-1}(1)(W_\alpha)$. One term in the determinant expansion of $M_n(0)(W_{\alpha(x)})$ by the first row is $1 \cdot x^2 M_{n-1}(1)(W_\alpha)$, and note that this is positive for any $x > 0$ and has order $2n$ in x. Any other term in the determinant expansion of $M_n(0)(W_{\alpha(x)})$ by the first row is of order $2n+2$ in x, and so we may make $M_n(0)(W_{\alpha(x)})$ positive by taking x sufficiently small, which completes the argument that $W_{\alpha(x)}$ has property $H(n)$. □

4. Flatness

We now consider the flatness of weighted shifts with property $H(n)$. As we discussed in the introduction, if W_α is subnormal (even 2-hyponormal) with $\alpha_n = \alpha_{n+1}$ $(n \in \mathbb{Z}_+)$, then $\alpha_1 = \alpha_2 = \cdots$. But this flatness property need not hold in weighted shifts with property $H(2)$ as we show next.

Example 4.1. Let $\alpha(x)$ be given by

$$\sqrt{x}, \sqrt{31/17}, \sqrt{31/17}, \sqrt{31/17}, \sqrt{31/17}, \sqrt{65/31}, \alpha_n = \sqrt{\frac{2^{n+1} + (-1)^{n+1}}{2^n + (-1)^n}} \ (n \geq 6).$$

(Recall that the tail of this sequence arises from $\mu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_2$ as in Example 2.2 and therefore $W_\alpha|_{\vee\{e_i\}_{i=4}^{\infty}}$ is of Hamburger-type.) One computes that $\det M_1(0) = \gamma_0\gamma_1(31/17 - x)$, $\det M_2(0) = 0$, $\det M_1(2) = 0$, and $\det M_2(2) = 0$. Positivity of other $M_1(2k)$ and $M_2(2k)$ is ensured because we have a Hamburger-type tail. Observe that $W_{\alpha(x)}$ has property $H(2)$ if and only if $0 < x \leq 31/17$. Thus, a weighted shift with property $H(2)$ may have five equal (successive) weights without being flat.

It turns out that property $H(n)$ for any $n \geq 3$ is sufficient to guarantee flatness if the two successive equal weights begin at an even index.

Theorem 4.2. *Let W_α be a weighted shift with property $H(3)$. If $\alpha_{2n} = \alpha_{2n+1}$ for some $n \in \mathbb{Z}_+$, then $\alpha_1 = \alpha_2 = \cdots$.*

Proof. Observe first that since the shift corresponding to the weight sequence $x, 1, 1, \ldots$ is subnormal for any $0 \leq x \leq 1$, it is impossible to extend flatness to include α_0.

Since $M_2(2k) \geq 0$ for all $k \in \mathbb{Z}_+$, by using the condition $\alpha_{2n} = \alpha_{2n+1}$, we obtain

$$\begin{aligned}\det M_2(2n) &= \gamma_{2n}\gamma_{2n+1}\gamma_{2n+2}\begin{vmatrix} 1 & \alpha_{2n}^2 & \alpha_{2n}^2\alpha_{2n+1}^2 \\ 1 & \alpha_{2n+1}^2 & \alpha_{2n+1}^2\alpha_{2n+2}^2 \\ 1 & \alpha_{2n+2}^2 & \alpha_{2n+2}^2\alpha_{2n+3}^2 \end{vmatrix} \\ &= -\gamma_{2n}\gamma_{2n+1}\gamma_{2n+2}\alpha_{2n+1}^2(\alpha_{2n+2}^2 - \alpha_{2n+1}^2)^2 \geq 0,\end{aligned}$$

where $|\cdot|$ denotes the determinant of a matrix, so obviously, $\alpha_{2n+1} = \alpha_{2n+2}$. Also, since $M_3(2k) \geq 0$ for all $k \in \mathbb{Z}_+$, by using the condition $\alpha_{2n} = \alpha_{2n+1} = \alpha_{2n+2}$, we get

$$\begin{aligned}\det M_3(2n) &= \gamma_{2n}\gamma_{2n+1}\gamma_{2n+2}\gamma_{2n+3}\begin{vmatrix} 1 & \alpha_{2n}^2 & \alpha_{2n}^2\alpha_{2n+1}^2 & \alpha_{2n}^2\alpha_{2n+1}^2\alpha_{2n+2}^2 \\ 1 & \alpha_{2n+1}^2 & \alpha_{2n+1}^2\alpha_{2n+2}^2 & \alpha_{2n+1}^2\alpha_{2n+2}^2\alpha_{2n+3}^2 \\ 1 & \alpha_{2n+2}^2 & \alpha_{2n+2}^2\alpha_{2n+3}^2 & \alpha_{2n+2}^2\alpha_{2n+3}^2\alpha_{2n+4}^2 \\ 1 & \alpha_{2n+3}^2 & \alpha_{2n+3}^2\alpha_{2n+4}^2 & \alpha_{2n+3}^2\alpha_{2n+4}^2\alpha_{2n+5}^2 \end{vmatrix} \\ &= \gamma_{2n}\gamma_{2n+1}\gamma_{2n+2}\gamma_{2n+3}\alpha_{2n+2}^6\left(\alpha_{2n+2}^2 - \alpha_{2n+3}^2\right)^3.\end{aligned}$$

Since $\det M_2(2n) = 0$, it follows from [6, Prop. 2.6] that $\det M_3(2n) = 0$. So $\alpha_{2n+2} = \alpha_{2n+3}$, i.e., $\alpha_{2n} = \alpha_{2n+1} = \alpha_{2n+2} = \alpha_{2n+3}$. Since $W_\alpha|_{\vee\{e_i\}_{i=2n+2}^\infty}$ is a weighted shift with property $H(3)$ such that $\alpha_{2n+2} = \alpha_{2n+3}$, by repeating the above method, we get $\alpha_{2n+2} = \alpha_{2n+3} = \alpha_{2n+k}$ for all $k \geq 4$.

On the other hand, since $M_2(2k-2) \geq 0$ for all $k \in \mathbb{N}$, by using the condition $\alpha_{2n} = \alpha_{2n+1}$ in the hypothesis, we get

$$\begin{aligned}\det M_2(2n-2) &= \gamma_{2n-2}\gamma_{2n-1}\gamma_{2n} \begin{vmatrix} 1 & \alpha_{2n-2}^2 & \alpha_{2n-2}^2\alpha_{2n-1}^2 \\ 1 & \alpha_{2n-1}^2 & \alpha_{2n-1}^2\alpha_{2n}^2 \\ 1 & \alpha_{2n}^2 & \alpha_{2n}^2\alpha_{2n+1}^2 \end{vmatrix} \\ &= -\gamma_{2n-2}\gamma_{2n-1}\gamma_{2n}\alpha_{2n-2}^2(\alpha_{2n}^2 - \alpha_{2n-1}^2)^2 \geq 0,\end{aligned}$$

which implies obviously that $\alpha_{2n-1} = \alpha_{2n}$. If n is 1, we stop here. Suppose $n \geq 2$. Since $M_3(2k-4) \geq 0$ for all $k \in \mathbb{N} \setminus \{1\}$, via the condition $\alpha_{2n-1} = \alpha_{2n} = \alpha_{2n+1}$, we get

$$\begin{aligned}\det M_3(2n-4) &= \gamma_{2n-4}\gamma_{2n-3}\gamma_{2n-2}\gamma_{2n-1} \begin{vmatrix} 1 & \alpha_{2n-4}^2 & \alpha_{2n-4}^2\alpha_{2n-3}^2 & \alpha_{2n-4}^2\alpha_{2n-3}^2\alpha_{2n-2}^2 \\ 1 & \alpha_{2n-3}^2 & \alpha_{2n-3}^2\alpha_{2n-2}^2 & \alpha_{2n-3}^2\alpha_{2n-2}^2\alpha_{2n-1}^2 \\ 1 & \alpha_{2n-2}^2 & \alpha_{2n-2}^2\alpha_{2n-1}^2 & \alpha_{2n-2}^2\alpha_{2n-1}^2\alpha_{2n}^2 \\ 1 & \alpha_{2n-1}^2 & \alpha_{2n-1}^2\alpha_{2n}^2 & \alpha_{2n-1}^2\alpha_{2n}^2\alpha_{2n+1}^2 \end{vmatrix} \\ &= -\gamma_{2n-4}\gamma_{2n-3}\gamma_{2n-2}\gamma_{2n-1}\alpha_{2n-4}^2\alpha_{2n-3}^4(\alpha_{2n-1}^2 - \alpha_{2n-2}^2)^3.\end{aligned}$$

Again using [6, Prop. 2.6] together with the fact $\det M_2(2n-2) = 0$, we get $\det M_3(2n-4) = 0$, so $\alpha_{2n-1} = \alpha_{2n-2} = \alpha_{2n} = \alpha_{2n+1}$. Repeat this processes until we get the equality $\alpha_1 = \cdots = \alpha_{2n}$. $\square$

Observe that in the work above we have actually proved along the way the following limited "propagation" result (and compare Example 4.1).

Corollary 4.3. *Let W_α be a weighted shift with property $H(2)$. If $\alpha_{2n} = \alpha_{2n+1}$ for some $n \in \mathbb{Z}_+$, then $\alpha_{2n-1} = \alpha_{2n} = \alpha_{2n+1} = \alpha_{2n+2}$.*

We leave to the interested reader the formulation of the results analogous to Theorem 4.2 and Corollary 4.3 in their versions for the properties $\widetilde{H}(n)$. These follow easily upon noting that if W_α has some property $\widetilde{H}(n)$, then the restriction $W_\alpha|_{\vee\{e_i\}_{i=1}^\infty}$ has property $H(n)$. Observe also that the combination of properties $H(2)$ and $\widetilde{H}(2)$ is equivalent to 2-hyponormality, and thus we may recapture the result of Curto in [5] from the two limited left- and right-propagation results.

It is natural to ask what propagation results, if any, arise from the combination of some property $H(n)$ and $\alpha_{2k-1} = \alpha_{2k}$. The next theorem shows that property $H(n)$ does not yield (further) flatness for any n.

Theorem 4.4. *Let $\alpha(x)$ be a weight sequence given by*

$$\alpha(x) : \sqrt{x}, \sqrt{2/3}, \sqrt{2/3}, \sqrt{4/5}, \alpha_k = \sqrt{\frac{k+1}{k+2}} \quad (k \geq 4)$$

with x a positive real variable. Suppose $n \geq 3$. Then there exist $\delta_n \in (0, 2/3)$ with $\delta_3 \geq \delta_4 \geq \cdots$ such that the weighted shift $W_{\alpha(x)}$ has property $H(n)$ for any $x \in (0, \delta_n]$ but does not have property $H(n)$ for any $x \in (\delta_n, 2/3]$.

Proof. Observe that

$$F_n(x) := \frac{9}{16x}\det M_n(0) = \begin{vmatrix} \frac{9}{16x} & \frac{9}{16} & \frac{3}{8} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \frac{9}{16} & \frac{3}{8} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+2} \\ \frac{3}{8} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots & \frac{1}{n+3} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+2} & \frac{1}{n+3} & \cdots & \cdots & \frac{1}{2n+1} \end{vmatrix}$$

$$= \frac{1}{x}\frac{9}{16}\begin{vmatrix} \frac{3}{8} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+2} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots & \frac{1}{n+3} \\ \frac{1}{5} & \frac{1}{6} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{1}{n+2} & \frac{1}{n+3} & \cdots & \cdots & \frac{1}{2n+1} \end{vmatrix} + \begin{vmatrix} 0 & \frac{9}{16} & \frac{3}{8} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \frac{9}{16} & \frac{3}{8} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+2} \\ \frac{3}{8} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots & \frac{1}{n+3} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+2} & \frac{1}{n+3} & \cdots & \cdots & \frac{1}{2n+1} \end{vmatrix}$$

$$=: \frac{1}{x}a_n + b_n.$$

By a direct computation with the determinant formula (3.1), we can see easily $a_n > 0$, indeed,

$$a_n = \frac{9}{16}\left(288 + n^2(n+1)^2(n+2)^2\right)\left(\prod_{k=1}^{n-1} k!\right)^3 n!(n+1)!\left(288\prod_{k=n+2}^{2n+1} k!\right)^{-1}.$$

This implies that $F_n(x) \geq 0$ on some interval $(0, \varepsilon_n)$. Put

$$f_n := \sup\{x \in (0, 2/3) : F_n(x) \geq 0\}, \quad n \geq 3.$$

Then set $\delta_n = \min\{f_i : i \leq n\}$ for $n \geq 3$. If $x < \delta_n$, then each $F_i(x)$ is strictly positive, and it follows easily from the Nested Determinant Test that $W_{\alpha(x)}$ has property $H(n)$; then $W_{\alpha(x)}$ has the property $H(n)$ on the interval $(0, \delta_n]$ since positivity varies continuously with x. Obviously, $\delta_3 \geq \delta_4 \geq \cdots$, and these are the required numbers. □

Remark 4.5. For our convenience, we record that

$$\delta_n = \sup\{x \in (0, 2/3] : W_{\alpha(x)} \text{ has property } H(n)\}, \quad n \in \mathbb{N},$$

where δ_n are as in the proof of Theorem 4.4. By direct computation with [13], we may check $\delta_1 = \delta_2 = \frac{2}{3} > \delta_3 = \frac{18}{55}$, and $\delta_4 = \frac{850}{10459}$, etc., and obtain easily the values δ_n for low numbers n, for example, $n = 3, 4, \ldots, 20$, etc. The Cauchy determinant formula (3.1) provides a good information to estimate the exact values δ_n for $n \in \mathbb{N}$ and the limit of the sequence $\{\delta_n\}_{n=1}^{\infty}$. We leave their computations of values in n to interested readers. We note also that it appears that $\delta_n = f_n$ (that is, the positivity of the matrix is in fact driven by the positivity of its determinant and not those of submatrices) but we are unable to show this in general.

5. Backward extensions and perturbations

Suppose W_α is a Hamburger-type weighted shift. Let $\alpha(x) : x, \alpha_0, \alpha_1, \cdots$ be a backward 1-step extension of the weight sequence α. It turns out as we see next that such a "backward 1-step" extension is not, perhaps, the natural thing to study.

Proposition 5.1. *Suppose W_α is a Hamburger-type weighted shift such that for some $x > 0$, $W_{\alpha(x)}$ is a Hamburger-type weighted shift. Then W_α is subnormal. In this case, any backward 1-step extension $W_{\alpha(x)}$ of W_α is a Hamburger-type shift if and only if $W_{\alpha(x)}$ is subnormal.*

Proof. Suppose that for some $x > 0$, $W_{\alpha(x)}$ is a Hamburger-type weighted shift. It is easy to see that for any $n \in \mathbb{N}$ and $k \in \mathbb{Z}_+$, $M_n(2k+1)W_{\alpha(x)} = x^2 M_n(2k)(W_\alpha)$ and therefore will be positive since W_α has property $H(n)$. It follows that $W_{\alpha(x)}$ has property $\widetilde{H}(\infty)$, and since by assumption it has property $H(\infty)$, it is subnormal. Therefore W_α is subnormal, since it is a restriction of $W_{\alpha(x)}$. The remaining assertion is then obvious. $\square$

Note that the proof of Proposition 5.1 shows that if W_α has property $H(n)$ (respectively, is Hamburger-type), then any backward 1-step extension has property $\widetilde{H}(n)$ (respectively, has property $\widetilde{H}(\infty)$). The converse is equally easy.

The same sort of approach yields the following proposition.

Proposition 5.2. *Let W_α be a Hamburger-type weighted shift with property $\widetilde{H}(n)$ for some $n \in \mathbb{N}$. Suppose $M_n(1)$ is strictly positive. Then there exists $x \in \mathbb{R}_+^0$ such that $W_{\alpha(x)}$ has property $\widetilde{H}(\infty)$ and $H(n)$.*

Proof. It is easy to see that it is enough to ensure the positivity of $M_n(0)(W_{\alpha(x)})$. Since $M_n(1)(W_\alpha)$ is positive, so is $M_{n-1}(1)(W_\alpha)$, and it suffices to take x in $\mathbb{R}_+^0$ small enough as in the proof of Proposition 3.3. $\square$

The results above impel us to study backward extensions of "even" length. Consider now a backward extension of length two: W_α with $\alpha: \ \alpha_0, \alpha_1, \ldots$ and $\alpha(x,y): \ x, y, \alpha_0, \alpha_1, \ldots$ yielding the corresponding weighted shift $W_{\alpha(x,y)}$. The results here then work in any determinate case (for an unbounded densely defined shift) but it may be interesting to consider extensions in the indeterminate case. The following is completely parallel to portions of [10, Lemma 2.1], and see also [5, Prop. 8].

Theorem 5.3. *Suppose W_α is a Hamburger-type weighted shift with $\alpha = \{\alpha_i\}_{i=0}^\infty$. Let $\alpha(x,y): \ x, y, \alpha_0, \alpha_1, \cdots$ be a backward 2-step extension of α, where $x, y \in \mathbb{R}_+^0$. Then $W_{\alpha(x,y)}$ is a Hamburger-type weighted shift if and only if the following four conditions hold:*

$$\frac{1}{t^2} \in L^1(\mu), \ \int\limits_{\mathbb{R}} \frac{1}{t} d\mu(t) > 0, \ y = \frac{1}{\left(\int_{\mathbb{R}} \frac{1}{t} d\mu(t)\right)^{\frac{1}{2}}}, \ 0 < x \leq \left(\frac{\int_{\mathbb{R}} \frac{1}{t} d\mu(t)}{\int_{\mathbb{R}} \frac{1}{t^2} d\mu(t)}\right)^{\frac{1}{2}}, \tag{5.1}$$

where μ is the Hamburger measure associated with W_α. In this case, $W_{\alpha(x,y)}$ has Hamburger measure ν defined by

$$d\nu = \lambda\delta_0 + x^2y^2 \cdot \frac{1}{t^2} d\mu, \quad \textit{where} \ \ \lambda = 1 - x^2y^2 \int\limits_{\mathbb{R}} \frac{1}{t^2} d\mu.$$

Proof. We first show that if the conditions in (5.1) hold, $W_{\alpha(x,y)}$ has property $H(\infty)$ and ν is as claimed. Since $\frac{1}{t^2} \in L^1(\mu)$ and $1 \in L^1(\mu)$ (because $\int_{\mathbb{R}} 1 d\mu = 1$), it is easy to see that $\frac{1}{t} \in L^1(\mu)$. Set

$$y = \frac{1}{(\int_{\mathbb{R}} \frac{1}{t} d\mu(t))^{\frac{1}{2}}}. \tag{5.2}$$

Let $x \in \mathbb{R}$ be such that

$$0 < x \leq \left(\frac{\int_{\mathbb{R}} \frac{1}{t} d\mu(t)}{\int_{\mathbb{R}} \frac{1}{t^2} d\mu(t)} \right)^{\frac{1}{2}}.$$

Then clearly $x^2 y^2 \int_{\mathbb{R}} \frac{1}{t^2} d\mu \leq 1$. Set $\lambda = 1 - x^2 y^2 \int_{\mathbb{R}} \frac{1}{t^2} d\mu$. Consider the measure

$$d\nu = \lambda \delta_0 + x^2 y^2 \frac{1}{t^2} d\mu.$$

One checks easily that $\int_{\mathbb{R}} 1 d\nu = 1$, so ν is a probability measure. Further, using (5.2),

$$\int_{\mathbb{R}} t d\nu = \int_{\mathbb{R}} t \left(\lambda \delta_0 + x^2 y^2 \frac{1}{t^2} d\mu \right) = x^2 y^2 \int_{\mathbb{R}} \frac{1}{t} d\mu = x^2.$$

Also,

$$\int_{\mathbb{R}} t^2 d\nu = \int_{\mathbb{R}} t^2 \left(\lambda \delta_0 + x^2 y^2 \frac{1}{t^2} d\mu \right) = x^2 y^2.$$

Further,

$$\begin{aligned} \int_{\mathbb{R}} t^n d\nu &= \int_{\mathbb{R}} t^n \left(\lambda \delta_0 + x^2 y^2 \frac{1}{t^2} d\mu \right) \\ &= x^2 y^2 \int_{\mathbb{R}} t^{n-2} d\mu \\ &= x^2 y^2 \gamma_{n-2}(W_\alpha) = \gamma_n(W_{\alpha(x,y)}), \qquad n \geq 3. \end{aligned}$$

Thus ν is a Hamburger measure yielding the moments of $W_{\alpha(x,y)}$, so $W_{\alpha(x,y)}$ has property $H(\infty)$ and has the measure claimed (where as noted before we have uniqueness because $\|W_{\alpha(x,y)}\| < \infty$).

Note in passing that from Hölder's inequality and since μ is a probability measure,

$$\begin{aligned} \left(\int_{\mathbb{R}} \frac{1}{t} d\mu \right)^2 &\leq \left(\int_{\mathbb{R}} \frac{1}{t} \cdot 1 d\mu \right)^2 \\ &\leq \left[\left(\int_{\mathbb{R}} \frac{1}{t^2} \cdot 1 d\mu \right)^{\frac{1}{2}} \cdot \left(\int_{\mathbb{R}} 1^2 d\mu \right)^{\frac{1}{2}} \right]^2 \\ &\leq \int_{\mathbb{R}} \frac{1}{t^2} d\mu. \end{aligned}$$

Thus

$$\frac{\int_{\mathbb{R}} \frac{1}{t} d\mu}{\int_{\mathbb{R}} \frac{1}{t^2} d\mu} \leq \frac{1}{\int_{\mathbb{R}} \frac{1}{t} d\mu}.$$

Therefore the third condition forces $x \leq y$, which we know is required for even property $H(1)$ for the backward extension.

Suppose now that there exist x and y (positive) so that $W_{\alpha(x,y)}$ has property $H(\infty)$. Clearly

$$\gamma_k(W_\alpha) \cdot x^2 \cdot y^2 = \gamma_{k+2}(W_{\alpha(x,y)}), \qquad k \geq 2.$$

Also, there exists a Hamburger measure $\hat{\nu}$ so that

$$\gamma_{k+2}(W_{\alpha(x,y)}) = \int_{\mathbb{R}} t^{k+2} d\hat{\nu}(t).$$

Then

$$\frac{1}{x^2y^2} \int_{\mathbb{R}} t^{k+2} d\hat{\nu}(t) = \int_{\mathbb{R}} t^k d\mu(t), \qquad k \geq 0. \tag{5.3}$$

By uniqueness of solutions in our (bounded) setting thus implies

$$d\hat{\nu} = d\hat{\beta} + x^2y^2 \frac{1}{t^2} d\mu, \tag{5.4}$$

where $\hat{\beta}$ is some atomic measure with all moments zero except possibly the zeroth and first. Observe also that necessarily $\frac{1}{t^2} \in L^1(\mu)$ and $\mu(\{0\}) = 0$, which is the first condition. Note that if $\operatorname{supp} \hat{\beta} \not\subseteq \{0\}$, then $\int_{\mathbb{R}} t^2 d\hat{\beta} \neq 0$, which contradicts (5.3) and (5.4) taken together. Then $\hat{\nu}$ has the form

$$d\hat{\nu} = \lambda\delta_0 + x^2y^2 \frac{1}{t^2} d\mu, \tag{5.5}$$

where $\lambda \geq 0$ is such that $\hat{\nu}$ is a probability measure. Using that $\hat{\nu}$ captures the moments of $\alpha(x, y)$ and computing with (5.5),

$$x^2 = \int_{\mathbb{R}} t d\hat{\nu} = \int_{\mathbb{R}} t(\lambda\delta_0 + x^2y^2 \frac{1}{t^2} d\mu) = x^2y^2 \int_{\mathbb{R}} \frac{1}{t} d\mu.$$

Note that since $\frac{1}{t^2} \in L^1(\mu)$, and $1 \in L^1(\mu)$, one has $|\frac{1}{t}| \in L^1(\mu)$ and so $\frac{1}{t} \in L^1(\mu)$. Thus using $x \neq 0$, $1 = y^2 \cdot \int_{\mathbb{R}} \frac{1}{t} d\mu$, and we must have

$$\int_{\mathbb{R}} \frac{1}{t} d\mu > 0 \quad \text{and } y = \frac{1}{(\int_{\mathbb{R}} \frac{1}{t} d\mu)^{1/2}}. \tag{5.6}$$

Finally, from $\lambda \geq 0$ we have

$$x^2y^2 \int_{\mathbb{R}} \frac{1}{t^2} d\mu \leq 1,$$

which given (5.6) yields the last condition and we are done. $\square$

The following remark can be obtained from the proof of Theorem 5.3.

Remark 5.4. The weight y in Theorem 5.3 is completely and uniquely determined by α (or equivalently μ). Further, if $\lambda = 0$ (equivalently, $\hat{\nu}(\{0\}) = 0$) x is also uniquely determined at its maximum possible value.

We may generalize to longer backward extensions in a familiar way (cf., for example, Theorem 3.5 of [10]).

Theorem 5.5. *Let W_α be a Hamburger-type weighted shift with weight sequence $\alpha = \{\alpha_i\}_{i=0}^\infty$ and let μ be the associated Hamburger measure. Suppose $x_1, x_2, \ldots, x_{2n-1}$, and x_{2n} are positive. Then $W_{\alpha(x_{2n}, x_{2n-1}, \cdots, x_2, x_1)}$ is a Hamburger-type weighted shift if and only if the following conditions hold*

(i) $\frac{1}{t^{2n}} \in L^1(\mu)$, $\int_{\mathbb{R}} \frac{1}{t^{2j-1}} d\mu > 0$, $1 \le j \le n$,

(ii) $x_j = \left(\frac{\| \frac{1}{t^{j-1}} \|_{L^1(\mu)}}{\| \frac{1}{t^j} \|_{L^1(\mu)}} \right)^{\frac{1}{2}}$ *for* $1 \le j \le 2n-1$, *and*

(iii) $0 < x_{2n} \le \left(\frac{\| \frac{1}{t^{2n-1}} \|_{L^1(\mu)}}{\| \frac{1}{t^{2n}} \|_{L^1(\mu)}} \right)^{\frac{1}{2}}$.

Further, in this case the Hamburger measure ν_{-2n} for $W_{\alpha(x_{2n}, x_{2n-1}, \cdots, x_2, x_1)}$ is

$$d\nu_{-2n} = \lambda_{-2n}\delta_0 + x_1^2 \cdots x_{2n}^2 \cdot \frac{1}{t^{2n}} d\mu, \quad \textit{where } \lambda_{-2n} = 1 - x_1^2 \cdots x_{2n}^2 \int_{\mathbb{R}} \frac{1}{t^{2n}} d\mu.$$

Proof. (*Sketch*) We invoke Theorem 5.3 repeatedly. If the proposed extension exists and is Hamburger-type, view it as acting on a space with basis

$$e_{-2n}, e_{-2n+1}, \ldots, e_{-2}, e_{-1}, e_0, e_1, \ldots$$

in the obvious way. Since the extension is Hamburger-type, so is its restriction

$$W^{(-2)} := W_{\alpha(x_{2n}, x_{2n-1}, \cdots, x_2, x_1)}|_{\vee\{e_i\}_{i=-2}^\infty}. \tag{5.7}$$

It is clearly a backward 2-step extension of W_α and therefore we must have the conclusions of Theorem 5.3, including

$$x_1 = \frac{1}{\left(\int_{\mathbb{R}} \frac{1}{t} d\mu(t) \right)^{\frac{1}{2}}}.$$

Also, we know the measure associated with $W^{(-2)}$ is of the form

$$d\nu_{-2} = \lambda_{-2}\delta_0 + x_1^2 x_2^2 \, \frac{1}{t^2} d\mu.$$

But since $W^{(-4)}$ (defined naturally as in (5.7)) is a backward 2-step extension of $W^{(-2)}$, we must have $\frac{1}{t^2} \in L^1(\nu_{-2})$, and so necessarily $\lambda_{-2} = 0$ and also x_2 is set at its maximum value

$$x_2 = \left(\frac{\int_{\mathbb{R}} \frac{1}{t} d\mu(t)}{\int_{\mathbb{R}} \frac{1}{t^2} d\mu(t)} \right)^{\frac{1}{2}}.$$

Since $\frac{1}{t^2} \in L^1(\nu_{-2})$ and $\lambda_{-2} = 0$ we get easily $\frac{1}{t^4} \in L^1(\mu)$, and other conclusions are equally simple. Repeating the process, we achieve the conclusion. □

We next give an example of a Hamburger-type weighted shift which is not Hausdorff-type but which allows Hamburger-type backward extensions.

Example 5.6. Let us consider a measure of the form

$$d\mu_\epsilon = \epsilon\delta_{-\epsilon} + \chi_{[\epsilon,1]}(t)dt, \quad 0 < \epsilon < 1,$$

which is modified from (3.2). Since

$$\gamma_n = \int_{\mathbb{R}} t^n \, d\mu = (-1)^n \epsilon^{n+1} + \frac{1}{n+1}\left(1 - \epsilon^{n+1}\right), \qquad n \in \mathbb{Z}_+, \tag{5.8}$$

it is obvious that $\gamma_{2k} > 0$ for all $k \in \mathbb{Z}_+$. By (5.8), $\gamma_{2k+1} = \frac{2k+3}{2k+2}\left(\frac{1}{2k+3} - \epsilon^{2k+2}\right) > 0$ for any ϵ such that $0 < \epsilon < e^{-1}$. Let μ_ϵ be a moment measure with $0 < \epsilon < e^{-1}$ and let W_α be the associated weighted shift. Then W_α satisfies the four conditions of (5.1) in Theorem 5.3. But it does not satisfy Theorem 5.5(i); indeed, $\int t^{-3} \, d\mu_\epsilon = -\frac{1}{2}\left(\epsilon^{-2} + 1\right) < 0$. Hence W_α is Hamburger-type backward 2-step extendable but not a Hamburger-type backward 4-step extendable weighted shift.

We may now turn to perturbations. Theorem 5.3 shows that a non-zero perturbation in the weights α_0 and α_1 which yields a shift with property $H(\infty)$ must be, in fact, one fixing α_1, and decreasing α_0 (view W'_α as a backward 2-step extension of $W_\alpha|_{\vee\{e_i\}_{i=2}^\infty}$). What follows is the analogue of the rest of Theorem 2.1 of [8], and with a similar proof.

Theorem 5.7. *No finite perturbation of the weights of some Hamburger-type weighted shift W_α that actually changes some α_j, $j \geq 1$, can yield a Hamburger-type weighted shift $W_{\alpha'}$.*

Proof. The observation before this theorem shows that no perturbation limited to $\{\alpha_0, \alpha_1\}$ can do other than as claimed. Consider then some perturbation actually changing some α_j with $j \geq 2$ and call the resulting weight sequence α' and the resulting shift $W_{\alpha'}$, supposing the latter to have property $H(\infty)$. Choose k such that all weights $\alpha_{2k}, \alpha_{2k+1}, \ldots$ are left unchanged but one or both of α_{2k-1} and α_{2k-2} is changed (note $k \geq 2$). For any shift W_β we denote by $W_{\beta_m}^{[m]}$ the restriction of W_β to $\vee_{\{e_m, e_{m+1}, \cdots\}}$, and note that it has weights $\beta_m, \beta_{m+1}, \ldots$. If W_β is Hamburger-type, and if m is even, let $\mu^{[m]}$ be the associated Hamburger measure (which exists since $W_{\beta_m}^{[m]}$ is Hamburger-type as a restriction). Since $W_{\alpha'}$ has property $H(\infty)$, so does each $W_{\alpha'_{2p}}^{[2p]}$. Clearly $W_{\alpha'_{2k-2}}{}^{[2k-2]}$ is a backward 2-step extension of $W_{\alpha'_{2k}}^{[2k]}$ with weights $\alpha'_{2k-2}, \alpha'_{2k-1}, \ldots$, but by the choice of k we have $\alpha'_{2k} = \alpha_{2k}, \alpha'_{2k+1} = \alpha_{2k+1}, \ldots$. Then $W_{\alpha'_{2k-2}}{}^{[2k-2]}$ is a Hamburger-type backward 2-step extension of $W_{\alpha_{2k}}{}^{[2k]}$, and so its first weight is specified completely and must be α_{2k-1}. Further, since yet another backward extension $W_{\alpha'_{2k-4}}{}^{[2k-4]}$ is possible, citing Theorem 5.5 we must have $\mu^{[2k-2]}(\{0\}) = 0$. But then α'_{2k-2} is completely determined by $\{\alpha_{2k}, \alpha_{2k+1}, \cdots\}$ at its maximum value and using the last condition [with equality] one easily shows $\alpha'_{2k-2} = \alpha_{2k-2}$. But this contradicts the assumption that one of α_{2k-2} or α_{2k-1} was actually changed, and thus no such perturbation is possible. $\square$

6. A three weights completion problem

Let $\alpha_0, \alpha_1, \alpha_2$ be positive real numbers with $\alpha_0 < \alpha_1$. In this section we discuss a *Hamburger completion problem* with three weights $\alpha_0, \alpha_1, \alpha_2$ as the initial data: the goal is to find a weight sequence $\widehat{\alpha}$ extending $\alpha_0, \alpha_1, \alpha_2$ such that the associated weighted shift $W_{\widehat{\alpha}}$ is Hamburger-type. (Note that the restriction $\alpha_0 < \alpha_1$ is harmless, as $\alpha_0 \leq \alpha_1$ is forced by property $H(1)$, and if $\alpha_0 = \alpha_1$ then the flatness result in Theorem 4.2 forces all weights equal.) For this purpose, we consider two possibilities:

1° *the initial data give rise to a completion moment sequence which is Hamburger,*

2° *the initial data give rise to a completion moment sequence which is Hamburger with all positive moments.*

In the presence of 2° we may define a Hamburger-type weighted shift in the usual way (abiding by our assumption that weights are positive), but 1° is not enough for this. There are two approaches to trying to find some completion at least satisfying 1°, and we turn to the first, leaving the second for remarks at the end of the section.

We may imitate the Curto–Fialkow construction (see [6, p. 231]). This is most easily described in terms of weights, so assume $\alpha_0 < \alpha_1$ as above. (Note also $\alpha_2 \geq \alpha_1$ is the Stampfli case where a subnormal completion is possible.) Set

$$s_0 = \frac{\psi_1 - \sqrt{\psi_1^2 + 4\psi_0}}{2}, \qquad s_1 = \frac{\psi_1 + \sqrt{\psi_1^2 + 4\psi_0}}{2}, \qquad \text{and} \quad \rho = \frac{s_1 - \alpha_0^2}{s_1 - s_0}, \tag{6.1}$$

where

$$\psi_0 = -\frac{\alpha_0^2\alpha_1^2(\alpha_2^2 - \alpha_1^2)}{\alpha_1^2 - \alpha_0^2}, \qquad \psi_1 = \frac{\alpha_1^2(\alpha_2^2 - \alpha_0^2)}{\alpha_1^2 - \alpha_0^2}, \tag{6.2}$$

and define $\mu = \rho\delta_{s_0} + (1-\rho)\delta_{s_1}$. We will claim that μ yields the correct moments (hence at least initial weights) to match the initial data, and so it induces a Hamburger completion at least in the sense of 1°.

There are things to check (since we don't assume $\alpha_1 < \alpha_2$). First, $s_0, s_1 \in \mathbb{R}$, because the expression inside the defining square root is a quadratic in α_2^2 which, after the substitution $\alpha_1 = \alpha_0 + \epsilon$ with $\epsilon > 0$, has no real zeros. Since we know s_0 is real, and hence this quantity is positive for α_2 large enough (the Stampfli case), it must always be positive. So $s_0, s_1 \in \mathbb{R}$. (Note that we expect that for the Hamburger-type case s_0 may be negative, but this is in fact for us the case of interest.)

We also need ρ (real and) satisfying $0 \leq \rho \leq 1$. That ρ is real is easy.

For $\rho \geq 0$, one checks easily that $\rho(k\alpha_0, k\alpha_1, k\alpha_2) = k\rho(\alpha_0, \alpha_1, \alpha_2)$, where $\rho := \rho(\alpha_0, \alpha_1, \alpha_2)$ is as in (6.1). So it suffices to consider the case in which $\alpha_0 = 1$. Then after substituting $\alpha_1^2 = 1 + \epsilon$ (with $\epsilon > 0$) we must check

$$-1 + \alpha_2^2(1+\epsilon) - 3\epsilon + \sqrt{(1+\epsilon)(1 + 5\epsilon + 4\epsilon^2 + \alpha_2^4(1+\epsilon) - 2\alpha_2^2(1+3\epsilon))} > 0,$$

and via the usual technique of moving $-1 + \alpha_2^2(1+\epsilon) - 3\epsilon$ to the other side, squaring both sides, and simplifying, this turns out to be correct. For $\rho \leq 1$, we have $\rho = \frac{s_1 - \alpha_0^2}{s_1 - s_0}$ and we obviously need $\alpha_0^2 \geq s_0$. If $s_0 > 0$ we are in the Stampfli subnormal case and know $\rho \leq 1$; if $s_0 < 0$, obviously $\alpha_0^2 > s_0$. Therefore we may define the measure $\mu = \rho\delta_{s_0} + (1-\rho)\delta_{s_1}$.

By a direct computation, we see easily that the pair of conditions $p \leq q$ and $a < \frac{q}{p+q}$ in Lemma 6.3 below is equivalent to $\alpha_0 \leq \alpha_2 < \alpha_1$. According to Lemma 6.3, if $\alpha_2 < \alpha_0 < \alpha_1$, then γ_n can be negative for some $n \in \mathbb{N}$, which proves the "Moreover" part of Proposition 6.1.

Finally we arrive at the following.

Proposition 6.1. *Let $\alpha : \alpha_0, \alpha_1, \alpha_2$ be positive real numbers with $\alpha_0 < \alpha_1$. Then there exists a (2 atomic) measure $\mu = \rho\delta_{s_0} + (1-\rho)\delta_{s_1}$ with $0 \leq \rho \leq 1$, where ρ, s_0, s_1 are as in (6.1) and (6.2), and a sequence $\{\widehat{\gamma}_n\}_{n=0}^{\infty} \subset \mathbb{R}$ with $\widehat{\gamma}_j = \gamma_j$ $(j = 0, 1, 2)$ such that*

$$\widehat{\gamma}_n = \int_{\mathbb{R}} t^n \, d\mu, \qquad n \in \mathbb{Z}_+.$$

Moreover, if $\alpha_0 \leq \alpha_2 < \alpha_1$, we can take a sequence $\widehat{\alpha} = \{\widehat{\alpha}_n\}_{n=0}^{\infty} \subset \mathbb{R}$ with $\widehat{\alpha}_j = \alpha_j$ $(j = 0, 1, 2)$ such that $\widehat{\gamma}_n = \widehat{\alpha}_0^2 \cdots \widehat{\alpha}_{n-1}^2$ for $n \in \mathbb{Z}_+$ (cf. (6.3) *below*).

We remark that, in the case of "Moreover" part of Proposition 6.1, it is easy to verify that the weights satisfy the recursion

$$\widehat{\alpha}_n^2 = \psi_1 + \frac{\psi_0}{\widehat{\alpha}_{n-1}^2} \quad n \geq 1. \tag{6.3}$$

(One approach is to define $g_\alpha(t) = t^2 - \psi_1 t - \psi_0$, compute that its roots are s_0 and s_1, deduce from this that $\int_{\mathbb{R}} g_\alpha(t)\, d\mu(t) = 0$, and compute.)

Definition 6.2. Given initial positive weights $\alpha : \alpha_0 \leq \alpha_2 < \alpha_1$, we will denote the Hamburger completion sequence of weights arising via the construction captured in Proposition 6.1 by $(\alpha_0, \alpha_1, \alpha_2)^H$.

Note that in this case we do not allow $\alpha_0 < \alpha_1 < \alpha_2$ for which there is a (Hausdorff) Stampfli completion, nor do we allow $\alpha_0 < \alpha_1 = \alpha_2$ for which there is a (flat) Hausdorff completion, and recall that $\alpha_0 \leq \alpha_1$ is required by property $H(1)$.

The following computational lemma will give us what we need to determine when the moments resulting are positive.

Lemma 6.3. *Suppose $p, q > 0$ and $0 < a < 1$, and consider the measure $\mu := a\delta_{-p} + (1-a)\delta_q$. Then the moments γ_n are all positive if and only if $p \leq q$ and $a < \frac{q}{p+q}$.*

Proof. It is easy to check that the problem "scales" in the sense that if $c > 0$, the moments arising from $a\delta_{-cp} + (1-a)\delta_{cq}$ are positive exactly when those of μ are. Thus scaling by $\frac{1}{p}$ we may reduce to the case $p = 1$. But then the moments are

$$\gamma_n = a(-1)^n + (1-a)q^n$$

so the second of the desired inequalities follows from considering $n = 1$ while the first comes from considering n large. □

We may then obtain the following; with a slight abuse of previous language, we will say that a moment sequence has some property $H(n)$ with the obvious meaning.

Theorem 6.4. *Let $\alpha_0, \alpha_1, \alpha_2$ be positive real numbers. Then the condition $\alpha_0 \leq \alpha_2 < \alpha_1$ is equivalent to the assertion that the real numbers $\alpha_0, \alpha_1, \alpha_2$ produce a Hamburger completion $(\alpha_0, \alpha_1, \alpha_2)^H$ with strictly positive weights but whose associated weighted shift $W_{(\alpha_0,\alpha_1,\alpha_2)^H}$ is not subnormal.*

Proof. Suppose that $\alpha_0 \leq \alpha_2 < \alpha_1$. Then extension to a weight sequence with a Hausdorff moment sequence is impossible (it is well known that such a moment sequence requires weakly increasing weights). Then using Proposition 6.1 we may produce the completion $(\alpha_0, \alpha_1, \alpha_2)^H$ whose moment sequence is Hamburger. To show that such a moment sequence is all positive, we use Lemma 6.3, the definitions in (6.1) and (6.2), and some easy computations. Thus we can produce positive weights for $(\alpha_0, \alpha_1, \alpha_2)^H$.

Conversely, suppose that three initial weights $\alpha_0, \alpha_1, \alpha_2$ produce a Hamburger completion $(\alpha_0, \alpha_1, \alpha_2)^H$ satisfying the given conditions. By Lemma 6.3, we have $\alpha_0 \leq \alpha_2$. The inequality $\alpha_2 < \alpha_1$ follows from the discussion in Definition 6.2. □

Corollary 6.5. *Suppose $1 \leq y < x$. Then $(1, \sqrt{x}, \sqrt{y})^H$ has a backward 2-step Hamburger-type extension if and only if $y < \frac{2x-1}{x}$.*

Proof. The measure associated with $(1, \sqrt{x}, \sqrt{y})^H$ is $\mu = \rho\delta_{s_0} + (1-\rho)\delta_{s_1}$ as in Proposition 6.1, and in meeting the requirements of Theorem 5.3 the sole concern is to check

$$\int_{\mathbb{R}} \frac{1}{t} d\mu(t) > 0.$$

But direct computations (aided by $s_0 + s_1 = \psi_1$ and $s_0 s_1 = -\psi_0$) show that this latter condition is equivalent to $y < \frac{2x-1}{x}$. □

Before closing this paper, we discuss briefly the other approach to completing the initial data $\alpha_0, \alpha_1, \alpha_2$ (with $\alpha_0 < \alpha_1$) to yield a Hamburger sequence of moments.

Remark 6.6. Let $\alpha_0, \alpha_1, \alpha_2$ (with $\alpha_0 < \alpha_1$) be given as three initial weights. The following approach is to build "flat extensions" of

$$C_0 := \begin{pmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_2 \end{pmatrix}$$

in the sense of [7]. Under our assumptions C_0 is invertible, and define

$$\gamma_4 = -\frac{\gamma_2^2 - 2\gamma_1\gamma_2\gamma_3 + \gamma_3^2}{\gamma_1^2 - \gamma_2},$$

$$B_1 := \begin{pmatrix} \gamma_2 & \gamma_3 \\ \gamma_3 & x \end{pmatrix} = \begin{pmatrix} \gamma_2 & \gamma_3 \\ \gamma_3 & \gamma_4 \end{pmatrix} \quad \text{and} \quad W := C_0^{-1} B_1 = \begin{pmatrix} \varphi_0^1 & \varphi_0^2 \\ \varphi_1^1 & \varphi_1^2 \end{pmatrix}.$$

Define $B_j := C_{j-1} W$ and $C_j := B_j W$ recursively for $j \in \mathbb{N}$. Obviously, each B_j and C_j is symmetric. It turns out that with some computations we may produce successively larger flat extensions

$$A_{2n} = \begin{pmatrix} C_0 & B_1 & \cdots & C_n \\ B_1 & C_1 & \cdots & B_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ C_n & B_{n+1} & \cdots & C_{2n} \end{pmatrix} = \begin{pmatrix} \gamma_0 & \cdots & \gamma_{4n+1} \\ \vdots & \ddots & \vdots \\ \gamma_{4n+1} & \cdots & \gamma_{8n+2} \end{pmatrix}, \quad n \in \mathbb{N}_0,$$

and

$$A_{2n+1} = \begin{pmatrix} C_0 & B_1 & \cdots & B_{n+1} \\ B_1 & C_1 & \cdots & C_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n+1} & C_{n+1} & \cdots & C_{2n+1} \end{pmatrix} = \begin{pmatrix} \gamma_0 & \cdots & \gamma_{4n+3} \\ \vdots & \ddots & \vdots \\ \gamma_{4n+3} & \cdots & \gamma_{8n+6} \end{pmatrix}, \quad n \in \mathbb{N}_0,$$

of C_0. Since $\operatorname{rank} A_{2j} = \operatorname{rank} A_{2j+1} = \operatorname{rank} C_0$ for all j, the extensions A_n of C_0 are strictly positive. (The values γ_n are defined along the way in the process.) The resulting sequence $\{\gamma_n\}_{n=0}^{\infty}$ is a Hamburger sequence since each A_n is positive. The relationship between the approaches is that as part of the construction just mentioned we generate a recursion in the moments involving φ_0^1 and φ_1^1, and it is easy to compute both that $\varphi_0^1 = \psi_0$ and $\varphi_1^1 = \psi_1$ and to show the resulting recursion is the same as in (6.3). Thus the two extensions coincide, and we followed the first approach since it yields the relevant Hamburger measure directly.

Acknowledgments

The third author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (NRF-2012R1A2A2A02008590). The fourth author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2010-0025830). Some portion of this work was completed when the first author visited Kyungpook National University, and he wishes to thank the Mathematics Department for its kind hospitality.

References

[1] G. Adams, G. Exner, k-hyponormality and n-contractivity for Agler-type shifts, J. Operator Theory 71 (2014) 585–600.
[2] N. Akhieser, The Classical Moment Problem and Some Related Questions in Analysis, Oliver and Boyd, 1965.
[3] C. Berg, Fibonacci numbers and orthogonal polynomials, Arab J. Math. Sci. 17 (2011) 75–88.
[4] J. Bram, Subnormal operators, Duke Math. J. 22 (1955) 75–94.
[5] R. Curto, Quadratically hyponormal weighted shifts, Integral Equations Operator Theory 13 (1990) 49–66.
[6] R. Curto, L. Fialkow, Recursively generated weighted shifts and the subnormal completion problem, Integral Equations Operator Theory 17 (1993) 202–246.
[7] R. Curto, L. Fialkow, Recursively generated weighted shifts and the subnormal completion problem, II, Integral Equations Operator Theory 18 (1994) 369–426.
[8] R. Curto, W.Y. Lee, k-hyponormality of finite rank perturbations of unilateral weighted shifts, Trans. Amer. Math. Soc. 357 (2005) 4719–4737.
[9] R. Curto, S.H. Lee, Quartically hyponormal weighted shifts need not be 3-hyponormal, J. Math. Anal. Appl. 314 (2006) 455–463.
[10] R. Curto, S.S. Park, k-hyponormality of powers of weighted shifts via Schur products, Proc. Amer. Math. Soc. 131 (2002) 2761–2769.
[11] P. Halmos, Normal dilations and extensions of operators, Summa Bras. Math. 2 (1950) 51–65.
[12] I.B. Jung, C. Li, A formula for k-hyponormality of backstep extensions of subnormal weighted shifts, Proc. Amer. Math. Soc. 129 (2001) 2343–2351.
[13] Mathematica Version 10, Wolfram Research, Inc., Champaign, IL, 2011.
[14] M. Reed, B. Simon, Fourier Analysis, Self-Adjointness (Methods of Modern Math. Physics, II), Academic Press, 1975.
[15] A. Shields, Weighted shift operators and analytic function theory, Math. Surveys 13 (1974) 49–128.
[16] B. Simon, The classical moment problem as a self-adjoint finite difference operator, Adv. Math. 137 (1998) 82–203.
[17] J. Stampfli, Which weighted shifts are subnormal, Pacific J. Math. 17 (1966) 367–379.
[18] J. Sylvester, On the relation between the minor determinants of linearly equivalent quadratic functions, Philos. Mag. Fourth Ser. 1 (1851) 295–305.